SAT-Based Approaches to Treewidth Computation: An Evaluation

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February 25, 2018
ICTAI 2014

Contributions of Work

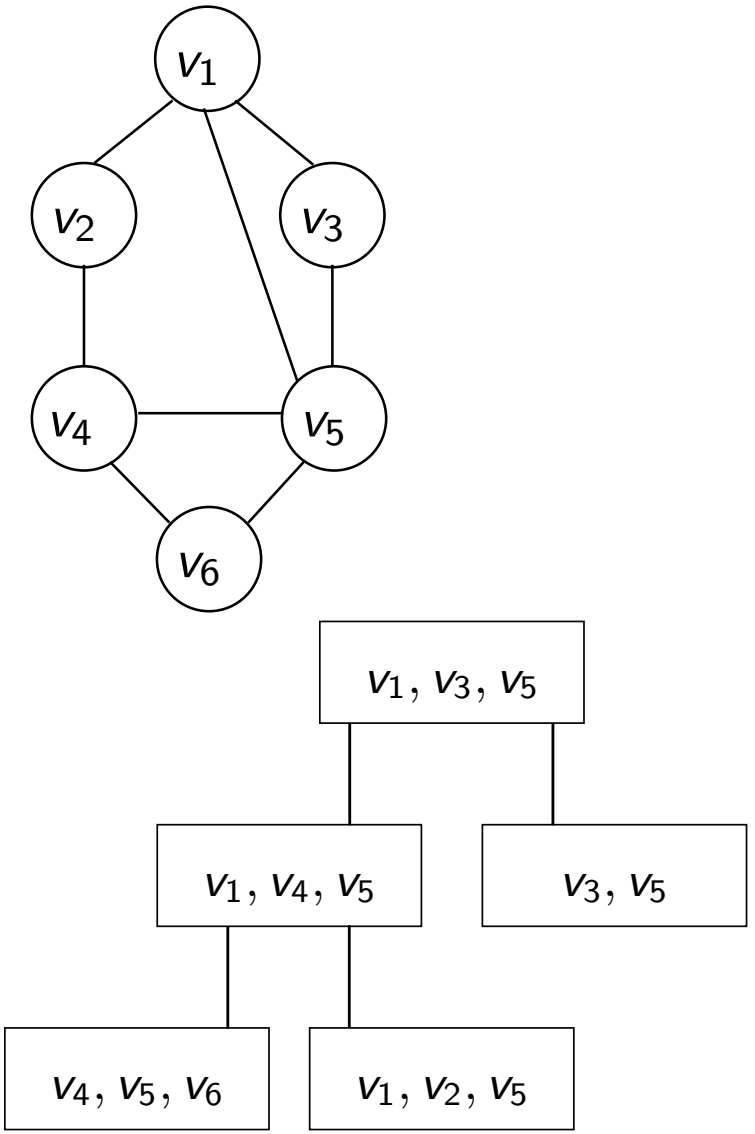
- Study Boolean Satisfiability (SAT) based methods for determining treewidth
  - Extend earlier studies Samer and Veith [2009]
- Experimental comparison:
  - Different SAT based methods
  - Specialized algorithms

Outline

- Treewidth
- Satisfiability
  - Iterative SAT
  - MaxSAT
- SAT based approach to Treewidth
- Experimental results
- Conclusion

Treewidth of an undirected graph G (TW)

- "How far" G is from being a tree
- $tw(G) \in \{1, \ldots, |V| - 1\}$
- Equivalently defined either in terms of tree-decompositions or linear-orders of the nodes



Formal Problem Statement

Given an undirected graph G, determine $tw(G)$

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Given an undirected graph G, determine $tw(G)$

- NP-Hard, (deciding $tw(G) \leq w$ is NP-complete)
- Previous work: hardness results, approximation algorithms

  Bodlaender and Koster [2011, 2010]; Amir [2001]; Becker and Geiger [1996]

- Exact Dynamic Programming and Branch and Bound algorithms

  Gogate and Dechter [2004]; Bodlaender et al. [2012]

Why important

- Fundamental, graph property
- Important connections to (in)tractability
  - Many algorithms for NP-hard problems exponential only in the treewidth:
    - Bucket Elimination, Junction tree Elimination
  - Several different NP-hard problems tractable when restricted onto bounded TW graphs:
    - Graph problems
    - Probabilistic inference
    - Constraint satisfaction

Note!

Optimal tree decomposition obtainable by our method

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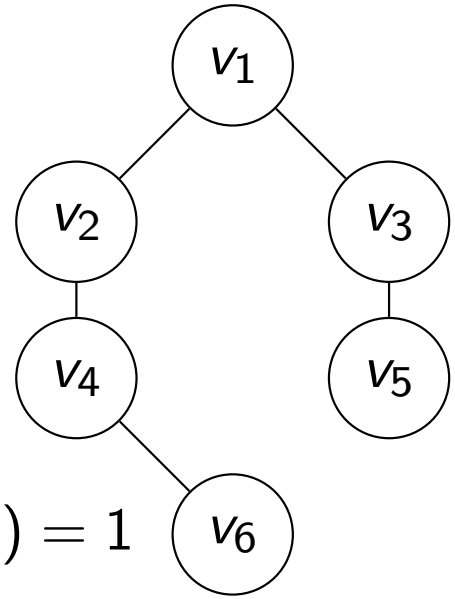
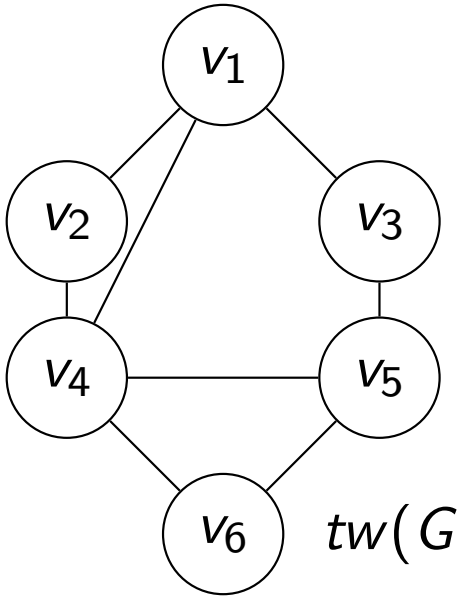
Treewidth Examples

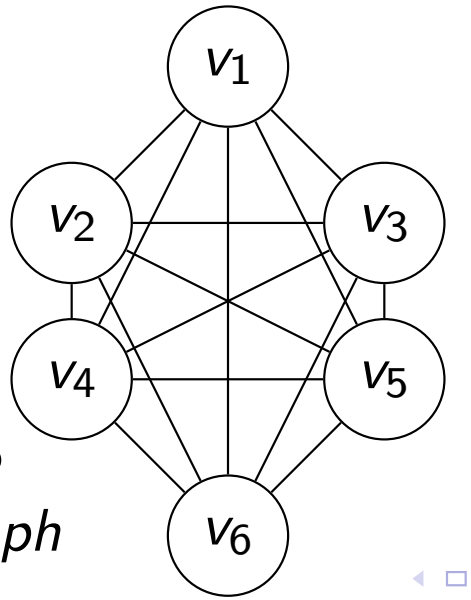


$tw(G) = 1$

tree



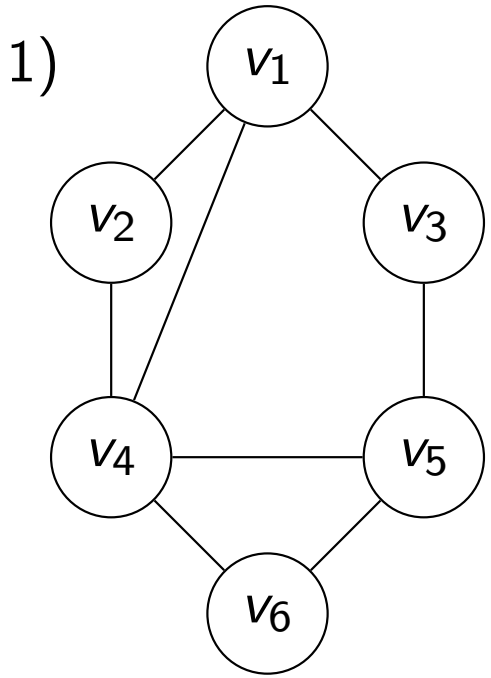
$tw(G) = 2$



$tw(G) = 5$

complete graph

Determining $tw(G)$



1. Fix an ordering: e.g. $v_6 \prec v_2 \prec v_4 \prec v_1 \prec v_3 \prec v_5$

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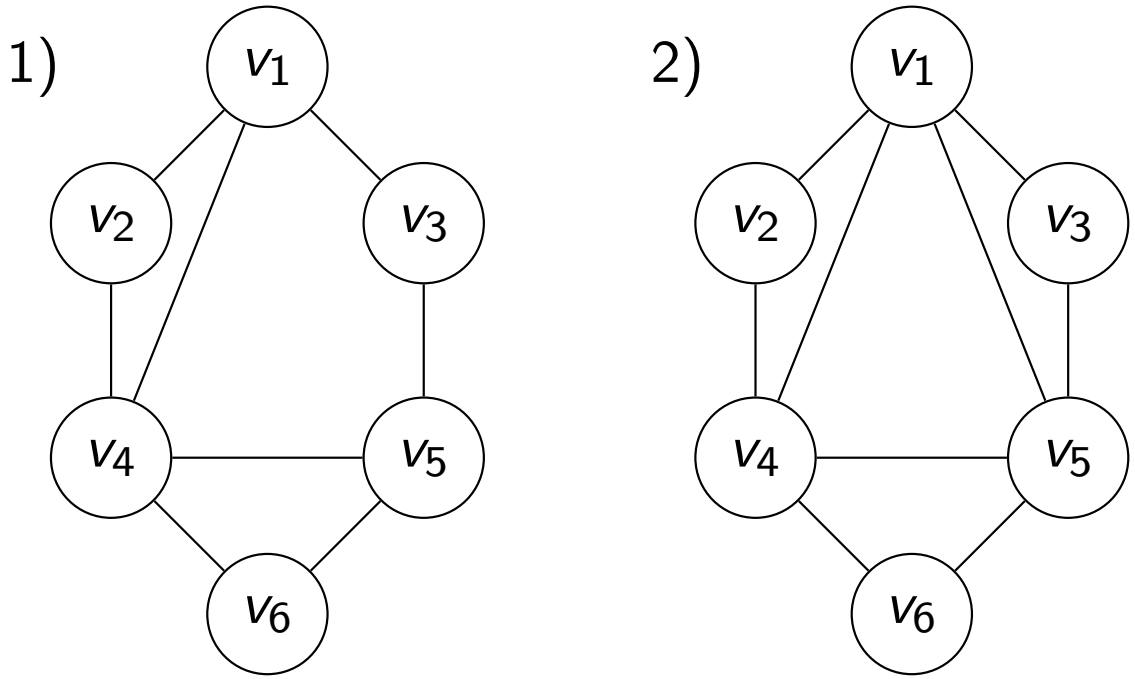

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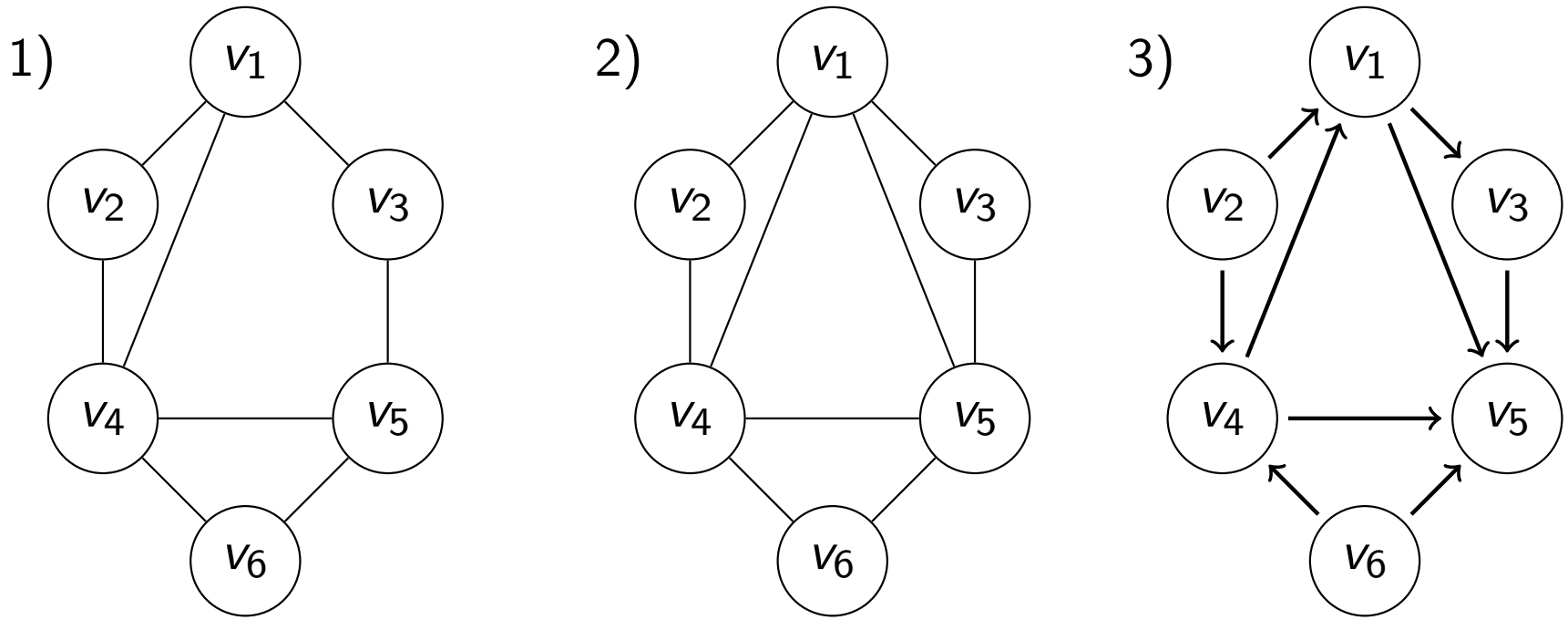

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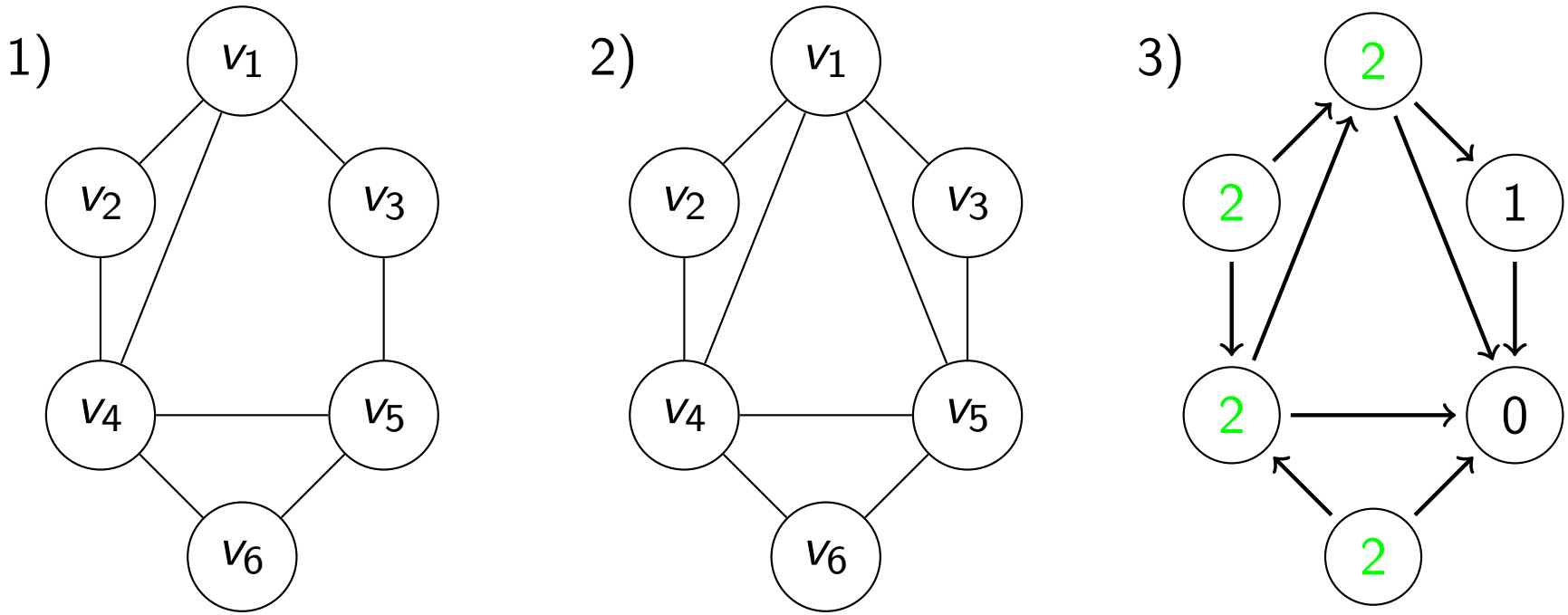
2. *Triangulate* w.r.t $\prec$
3. *Order* w.r.t. $\prec$ to form edges $\vec{\Delta}(E, \prec)$

Determining $tw(G)$



- Width of $\prec$: $\max_{v_i \in V} |\{(v_i, v_j) \in \vec{\Delta}(E, \prec)\}|$

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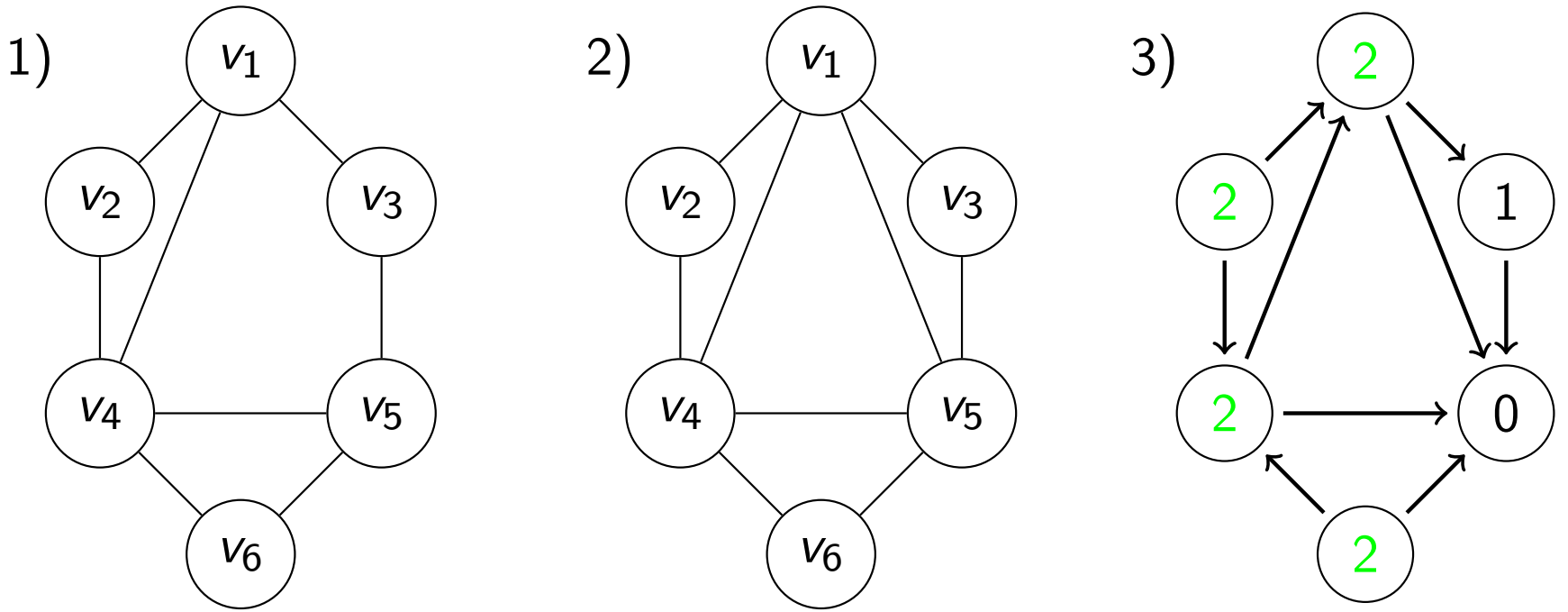

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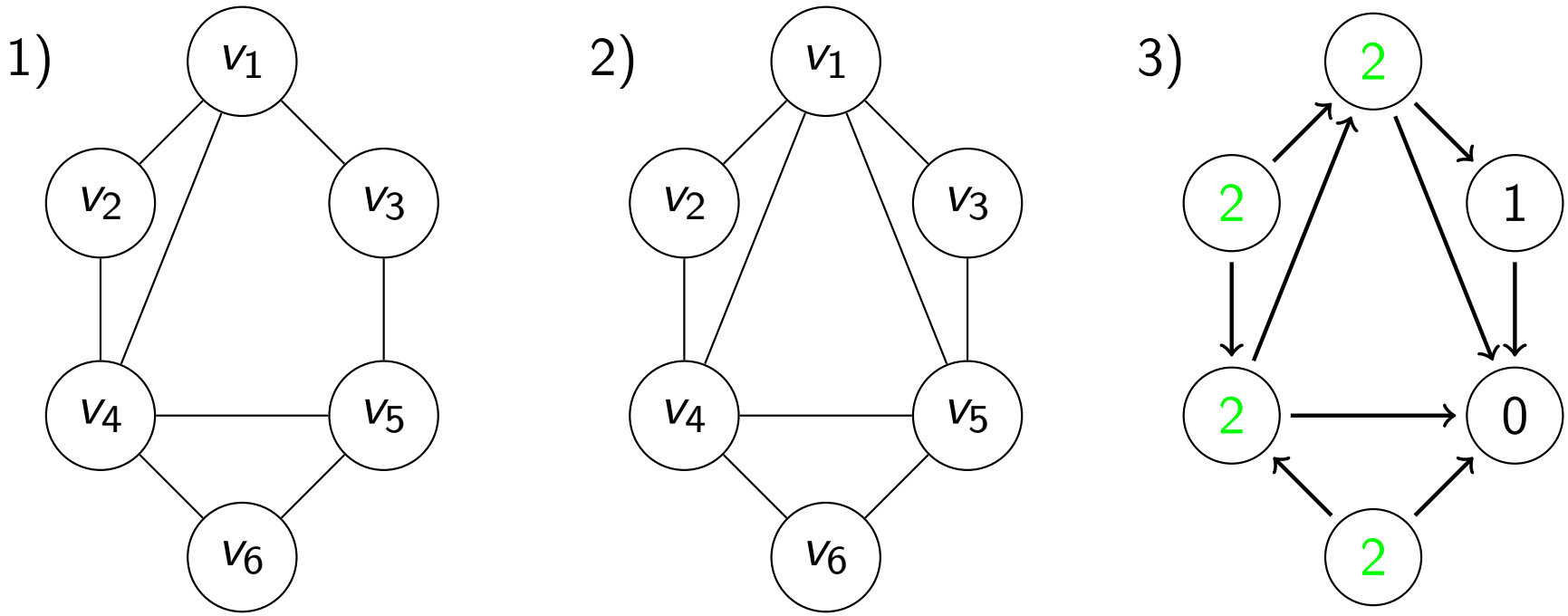

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NOTE: $tw(G) \leq k$ iff there exists $\prec$ of width $\leq k$

Satisfiability

- Literal: a boolean variable x or $\neg x$
- Clause C: a disjunction (set of) of literals
- Truth assignment τ: a function from boolean variables to $\{0, 1\}$
- $\tau(C) = 1$ if
  $\tau(x) = 1$ for a literal $x \in C$,
  $\tau(x) = 0$ for a literal $\neg x \in C$

The SAT problem

Given a CNF formula F (set of clauses), decide if F is satisfiable

F satisfiable iff there exists τ s.t. $\tau(C) = 1$ for all $C \in F$

Incremental SAT

- Many modern solvers offer incremental interface
- Allows querying with different assumptions without the need to recreate instance
  - Solver can retain information from previous queries

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- Well known optimization variant of the SAT problem
- Given two sets of clauses F_h, F_s find τ s.t.
  1. $\tau(C) = 1$ for all $C \in F_h$
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An optimal MaxSAT solution:
$\tau(x_3) = 1$, $\tau(x_1) = 0$ and $\tau(x_2) = 1$.

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Encoding of Treewidth into CNF

Basic Idea:

- Model ordered graph as clauses
- Constrain $|\{(v_i, v_j) \in \vec{\Delta}(E, \prec)\}| \leq k$ for all i
- Find smallest k for which the resulting CNF formula is satisfiable

Variables:

- $ord_{ij} = 1$ iff $v_i \prec v_j$
- $O_{ij} = 1$ iff $(v_i, v_j) \in \vec{\Delta}(G, \prec)$

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Modeling the ordered graph

- *ord* represent a linear order:
  $ord_{ij} \wedge ord_{jk} \rightarrow ord_{ik}$.
- Each edge in G corresponds to an edge in $\vec{\Delta}(G, ord)$:
  $(O_{ij} \vee O_{ji})$
- Common predecessor corresponds to an edge in $\vec{\Delta}(G, ord)$:
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- Edges in $\vec{\Delta}(G, ord)$ are consistent with $\prec$:
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"Orig" SAT approach for TW Samer and Veith [2009]

For each $w = 1 \ldots N-1$, solve

$$\mathcal{F}_{\text{base}}(G) \wedge \bigwedge_{i=1}^{N} \sum_{j \neq i} O_{ij} \leq w$$

Encode $\sum_{j \neq i} O_{ij} \leq w$ using *improved sequential counter*

Return smallest w for which the instance is satisfiable

Enabling Iterative SAT

- Let $C(i)$, be a *cardinality network*: $y_i^1, \ldots, y_i^{N-1}$
  - If $y_i^{k+1} = 0$ then $\sum_{i \neq j} O_{ij} \leq k$
- Add:

$$\mathcal{W}_w \leftrightarrow (\neg y_1^{w+1} \wedge \neg y_2^{w+1} \wedge \ldots \wedge \neg y_N^{w+1})$$

- Now: $\mathcal{F}_{\text{base}}(G) \wedge \bigwedge_{i=1}^{N} C(i)$ satisfiable under the assumption $\mathcal{W}_w = 1$ iff $tw(G) \leq w$

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"Iter" SAT approach for TW

Solve $\mathcal{F}_{\text{base}}(G) \wedge \bigwedge_{i=1}^{N} C(i)$ iteratively under different assumptions

Return smallest w for which $\mathcal{F}_{\text{base}}(G) \wedge \bigwedge_{i=1}^{N} \mathcal{C}_w(i)$ is satisfiable under the assumption $\mathcal{W}_w = 1$

MaxSAT

- $tw(G) \leq w$ implies $tw(G) \leq w'$ for all $w' > w$
- If $\mathcal{F}_{\mathrm{base}}(G) \wedge \bigwedge_{i=1}^{N} C(i)$ satisfiable under the assumption $\mathcal{W}_w = 1$, same holds for all assumptions $\mathcal{W}_{w'} = 1$

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MaxSAT approach for TW

Let: $\mathcal{F}_{\max}(G) = (F_h, F_s)$ where
$F_h = \mathcal{F}_{\text{base}}(G) \wedge \bigwedge_{i=1}^{N} C(A_i)$ and
$F_s = \{(\mathcal{W}_i) \mid i = 0..(N-1)\}$.

Return optimal cost of $\mathcal{F}_{\max}(G)$

Experimental Evaluation

- Different search strategies for the Orig and Iter approach:
  - **Orig-U:** Start with $w = 1$, and increase while unsatisfiable.
  - **Orig-D:** Start with $w = N - 2$, and decrease while satisfiable.
  - **Iter-Lin-U:** Start with $w = 1$ and increase while unsatisfiable.
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Solvers used

- Orig: Minisat 2.2 Eén and Sörensson [2004]
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- MaxSAT: WMaxSatz09, MSUnCore, MaxHS and OpenWBO

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Experimental Evaluation

Benchmarks

- Real world graph coloring
- UCL Bayesian networks
- Benchmarks from Samer and Veith [2009]
- Number of nodes $11 \ldots 128$
- Real treewidth $5 \ldots 33$

Results, Overview

Search Strategies using MiniSAT

- All search strategies solved 31 instances.
- Time spent by different strategies:
  - Orig-D: 22911s, Orig-U 13252s
  - Iter-Lin-U 4543s, Iter-Lin-D 2634s, Iter-Lin-Bin 3610s

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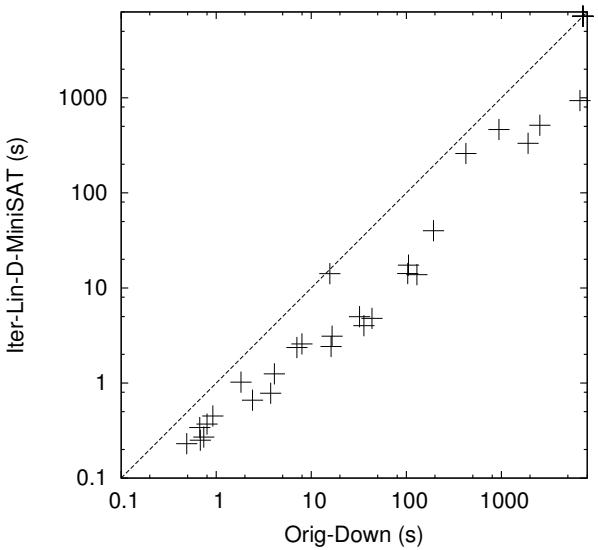
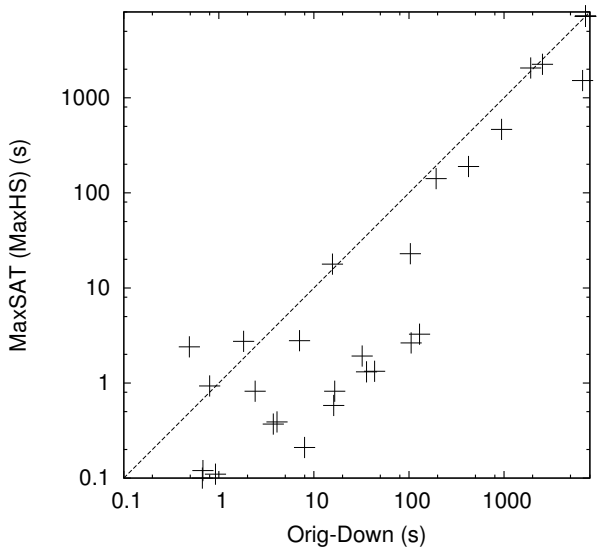
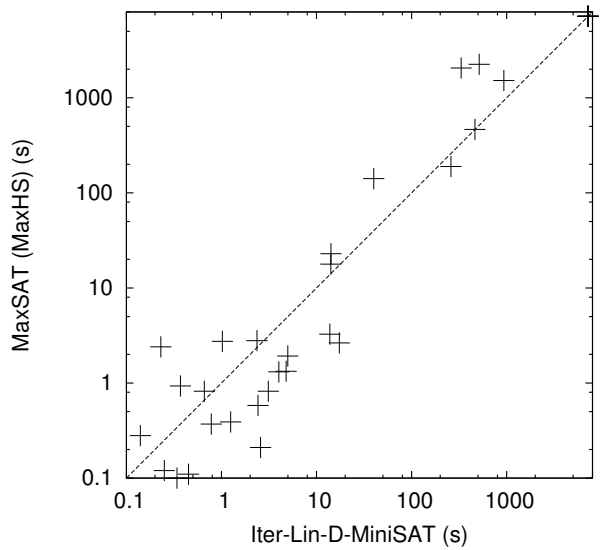
Solvers

- Both Glucose and MiniSat solved 31 instances.
  MiniSAT in 3610s, Glucose in 6003s
- Instances solved by MaxSAT:
  - MaxHS, Open Wbo: 31.
    - MaxHS in 6686s, Open Wbo in 7182s.
  - MSUnCore: 29
  - WMaxSatz: 3

Results


Iter-Lin-D-MiniSAT (s)
Orig-Down (s)
0.1
1
10
100
1000


MaxSAT (MaxHS) (s)
Orig-Down (s)
0.1
1
10
100
1000


MaxSAT (MaxHS) (s)
Iter-Lin-D-MiniSAT (s)
0.1
1
10
100
1000

Comparison to other algorithms

- Compare to previously proposed Dynamic Programming (DP) and Branch and Bound (Quick BB) algorithms

Bodlaender et al. [2012]; Gogate and Dechter [2004]

Instance	Orig-Mini-U	Iter-Mini-Bin	MaxSAT	QuickBB	DP
Myciel 4	15.69	10.53	17.80	0.09	4.95
Barley-pp	0.49	0.19	2.40	0.81	221.39
David PP	1.82	0.78	2.74	0.94	52.34
Ship-ship-pp	192.90	224.09	141.20	MEM	6291.38
Queen 6	TO	TO	TO	31.83	23.51
Alarm	0.74	0.31	0.08	0	MEM
oesoca-42	0.92	0.37	0.11	MEM	MEM
Eil51	6713.35	532.94	1517.53	MEM	MEM
Cellar09-pp	16.15	2.12	0.58	0	MEM
1en2	1910.56	470.56	2056.78	MEM	MEM
Hepar2	16.54	2.46	0.82	0	MEM
Mulsol.i.5.-pp	2539.76	1915.01	2251.63	0.04	MEM
Miles500	942.95	164.82	465.25	MEM	MEM

Comparison to other algorithms

- Compare to previously proposed Dynamic Programming (DP) and Branch and Bound (Quick BB) algorithms

  Bodlaender et al. [2012]; Gogate and Dechter [2004]

- DP often runs out of memory

Instance	Orig-Mini-U	Iter-Mini-Bin	MaxSAT	QuickBB	DP
Myciel 4	15.69	10.53	17.80	0.09	4.95
Barley-pp	0.49	0.19	2.40	0.81	221.39
David PP	1.82	0.78	2.74	0.94	52.34
Ship-ship-pp	192.90	224.09	141.20	MEM	6291.38
Queen 6	TO	TO	TO	31.83	23.51
Alarm	**0.74**	**0.31**	**0.08**	**0**	**MEM**
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Eil51	**6713.35**	**532.94**	**1517.53**	**MEM**	**MEM**
Cellar09-pp	**16.15**	**2.12**	**0.58**	**0**	**MEM**
1en2	**1910.56**	**470.56**	**2056.78**	**MEM**	**MEM**
Hepar2	**16.54**	**2.46**	**0.82**	**0**	**MEM**
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Comparison to other algorithms

- Compare to previously proposed Dynamic Programming (DP) and Branch and Bound (Quick BB) algorithms

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- Quick-BB fastest on several instances

Instance	Orig-Mini-U	Iter-Mini-Bin	MaxSAT	QuickBB	DP
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Comparison to other algorithms

- Compare to previously proposed Dynamic Programming (DP) and Branch and Bound (Quick BB) algorithms

  Bodlaender et al. [2012]; Gogate and Dechter [2004]

- Some instances best solved by SAT-based approaches

Instance	Orig-Mini-U	Iter-Mini-Bin	MaxSAT	QuickBB	DP
Myciel 4	15.69	10.53	17.80	0.09	4.95
Barley-pp	0.49	0.19	2.40	0.81	221.39
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Conclusions

- Experimentally compared 3 different SAT based approaches to determining $tw(G)$
  - Iterative SAT and MaxSAT outperform the independent approach.
- SAT-based approaches orthogonal to specialized algorithms in our tests.
- Viable approach for determining treewidth on moderately sized graphs.

Further queastions

- SAT-based methods as approximative algorithms.
- Other constraint languages.

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Thank you for your attention

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The original encoding

Does not include

- $(O_{ki} \wedge O_{kj}) \rightarrow (O_{ij} \vee O_{ji})$.
- $(O_{ij} \vee O_{ji})$
- $ord_{ij} \rightarrow \neg O_{ji}$

Includes

1. $ord_{ij} \rightarrow O_{ij}$ and $ord_{ji} \rightarrow O_{ji}$
2. $(O_{ki} \wedge O_{kj} \wedge ord_{ij}) \rightarrow O_{ij}$ and $(O_{ki} \wedge O_{kj} \wedge ord_{ji}) \rightarrow O_{ji}$.

Iterative SAT as anytime